spl(p,q) superalgebra and differential operators

Y. Brihaye

Department of Mathematical Physics

University of Mons

Av. Maistriau, B-7000 MONS, Belgium.

Stefan Giller*, Piotr Kosinski *

Department of Theoretical Physics

University of Lodz

Pomorska 149/153, 90-236 Lodz, Poland

Abstract

Series of finite dimensional representations of the superalgebras spl(p,q) can be formulated in terms of linear differential operators acting on a suitable space of polynomials. We sketch the general ingredients necessary to construct these representations and present examples related to spl(2,1) and spl(2,2). By revisiting the products of projectivised representations of sl(2), we are able to construct new sets of differential operators preserving some space of polynomials in two or more variables. In particular, this allows to express the representation of spl(2,1) in terms of matrix differential operators in two variables. The corresponding operators provide the

*† Work supported by grant n^o KBN 2P03B07610

building blocks for the construction of quasi exactly solvable systems of two and four equations in two variables. We also present a quommutator deformation of spl(2,1) which, by construction, provides an appropriate basis for analyzing the quasi exactly solvable systems of finite difference equations.

In memory of our friend S. Malinowski

1 Introduction

The number of quantum mechanical problems for which the spectral equation can be solved algebraically is rather limited. It is therefore not surprising that the quasi exactly solvable (QES) equations [1], [2] attract some attention. For these equations, indeed, a finite number of eigenvectors can be obtained by solving an algebraic equation. The study of QES equations has motivated the classification of finite dimensional real Lie algebras of first order differential operators. The algebras which, in a suitable representation, preserve a finite dimensional module of smooth real functions are particularly relevant for QES equations. The case of one variable was addressed and solved some years ago [3]. There is, up to an equivalence only one algebra, for instance sl(2), acting on the space of polynomials of degree at most n in the variable. For two variables, the classification is more involved. It is described respectively in [4] and [5] for complex and real variables. The corresponding real QES operators finally emerge in seven classes summarized in table 7 of [5].

The natural next step is to classify the QES systems of two equations [6, 7, 8]. Obviously, the number of possible finite dimensional modules of functions to be preserved grows with the number of independent variables admitted. It is known that, in general, the underlying algebraic structure is not a (super) Lie algebra. As far as 2×2 matrix differential operators of one variable are considered, the few cases where the algebra is a Lie algebra, it is generically sl(2)×sl(2) or spl(2,1) [7]. A generalisation of these operators to the case of V variables leads naturally to the symmetry algebras sl(2)×sl(V+1) or spl(V+1,1) [8]. The first case is rather trivial while the case where a superalgebra is involved seems to be worth to be

studied in more details. In the present paper we study the modules of polynomials is several variables which can serve as basis of representation to the superalgebras of the spl(p,q) family [9].

In the second section, we present some new aspects of the representations of the Lie algebra sl(p) in terms of differential operators. As a byproduct we obtain a simpler description of some of the QES operators classified in [5].

Thereafter, in section 3 we consider the superalgebra spl(p,q) and sketch the form of its representations which are expressible in terms of linear differential operators. We construct explicitely some of these representations in the cases spl(2,1) and spl(2,2). We obtain in particular a realization of spl(2,1) in terms of partial differential operators of two real variables. This provides the algebraic basis for the description of certain QES systems of two or four equations in two variables. The properties of the relevant tensorial operators are given in section 4.

The tensorial labelling used to write the generators of the algebras considered simplifies considerably the presentation of their structure constants. Taking advantage of this labelling, we are able to find a deformation of spl(2,1) similar, in the spirit, to the Witten-Woronowicz deformation of sl(2) [10, 11, 12]. For this deformed algebra, we also find the representations that we express in terms of finite difference operators. The normal ordering rules obeyed by these operators are more appropriate for the classification of finite difference QES operators than the ones given in a previous paper [13].

2 Lie algebra sl(M+1)

Consider M independent real variables $\vec{x} \equiv x_1, ..., x_M$ and denote $P(m, M)$ the vector space of polynomials of overal degree at most m in these variables i.e.

$$P(m, M) = \mathrm{span}\{x_1^{m_1} x_2^{m_2} \cdots x_M^{m_M} \quad ; \quad m_1 + m_2 + \cdots m_M \leq m\} \qquad (1)$$

(for shortness we use $P(m,1) \equiv P(m)$). The linear differential operators preserving $P(m,M)$ can be constructed (in a sense of enveloping algebra) out of $(M+1)^2$ operators [6, 8]

$$
\begin{aligned}
J_0^0(\vec{x},m) &= D - m + \gamma \quad , \quad D \equiv \sum_{j=1}^{M} x_j \frac{\partial}{\partial x_j} \\
J_0^k(\vec{x},m) &= \frac{\partial}{\partial x_k} \quad , \quad k = 1,\ldots,M \\
J_k^0(\vec{x},m) &= -x_k(D-m) \quad , \quad k = 1,\ldots,M \\
J_k^l(\vec{x},m) &= -x_k \frac{\partial}{\partial x_l} \quad + \gamma\delta_k^l \ , \quad k,l = 1,\ldots,M
\end{aligned} \tag{2}
$$

which obey the commutation relations of $gl(M+1)$:

$$
[J_a^b, J_c^d] = \delta_a^d J_c^b - \delta_c^b J_a^d \quad , \quad a,b,c,d = 0,1,\ldots,M \ . \tag{3}
$$

irrespectively of the constant γ.

For later convenience we introduce for $M=1$ the fundamental QES operators

$$
\begin{aligned}
j_+(x,m) &= -J_1^0(x,m) = x(\partial_x - m) \\
j_0(x,m) &= \frac{1}{2}(J_0^0(x,m) - J_1^1(x,m)) = \partial_x - \frac{m}{2} \\
j_-(x,m) &= -J_1^0(x,m) = \partial_x
\end{aligned} \tag{4}
$$

which represent the Lie algebra sl(2) [1].

The representation of $gl(M+1)$ defined above is irreducible and has dimension C_M^{M+m}, corresponding to the Young diagram with one line of M elementary boxes. Series of reducible representations of $gl(M+1)$ can trivially be obtained by considering the vector space

$$
P(m_1,M) \oplus P(m_2,M) \oplus \cdots \oplus P(m_k,M) \tag{5}
$$

and building the suitable direct sums of the operators above

$$
\mathrm{diag}(J_a^b(\vec{x},m_1), J_a^b(\vec{x},m_2), \cdots, J_a^b(\vec{x},m_k)) + \delta_a^b \mathrm{diag}(\gamma_1,\gamma_2,\cdots,\gamma_k) \tag{6}
$$

where we separate explicitely the parameters γ entering through (2).

2.1 Products of representations

Let $x_1, \dots, x_M$ and $y_1, \dots, y_N$ denote two sets of independent variables, we conveniently define $P(m, M; n, N)$ as the space of polynomials of overal degree at most m (resp. n) in the variables x_a (resp. y_b).

Another way to represent $gl(M+1)$ in terms of differential operators is to consider the generators corresponding to the product of two representations (2); that is to say,

$$J_a^b(\vec{x}, m; \vec{y}, n) = J_a^b(\vec{x}, m) + J_a^b(\vec{y}, n) \tag{7}$$

where $\vec{x}$ and $\vec{y}$ represent two sets of M independent variables. Clearly, the operators (7) obey the commutation relations (3). They act on the vector space $P(m, M; n, M)$. Their action is, however, not irreducible. For shortness, we discuss this statement in the case $M = 1$, $P(m, 1; n, 1)$ is abbreviated $P(m; n)$.

One can show easily that the operators (7) preserve irreducibly the subspace of $P(m; n)$ defined by

$$M(m; n) = \text{Span}\left\{ (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^k x^m y^n \quad , \quad 0 \leq k \leq m + n \right\} \tag{8}$$

This vector space is the eigenspace of the Casimir operator corresponding to the representation of highest spin in the decomposition of $P(m; n)$ in subspaces irreducible with respect to the action of (7). The other irreducible representations result as similar structures with different values of m and n; it is therefore sufficient to deal with (8).

Alternatively, $M(m; n)$ can be seen as the kernel of the operator

$$K = (x - y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + n\frac{\partial}{\partial x} - m\frac{\partial}{\partial y} \tag{9}$$

acting on $P(m; n)$. Remark that $M(m; 0)$ (resp. $M(0; n)$) is isomorphic to $P(m)$ (resp. $P(n)$); in that case, the one dimensional operators (4) are recovered from (7) by ignoring the partial derivative $\frac{\partial}{\partial y}$ (resp. $\frac{\partial}{\partial x}$).

2.2 QES operators in two variables

The QES operators in two real variables are classified in Ref. [5]; the authors summarize the seven possible hidden algebras in their table 7. The operators labelled (1.4), (1.10) and (2.3) in this table are studied independently in Refs. [6], [15]. The operators labelled (1.1) in the table appear to be new; in particular they lead to the only case for which the invariant module is not manifestly a space of polynomials in the two variables. In the following we show that the formulation (1.1) can be simplified and related to the algebra (7) by means of a suitable change of function.

The operators (1.1) in table 7 of ref.[5] read

$$\begin{aligned} \tilde{j}_- &= \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \\ \tilde{j}_0 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ \tilde{j}_+ &= x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + \frac{n}{2}(x-y) \end{aligned} \tag{10}$$

They preserve the space $\tilde{M}(m;n)$ defined as

$$\tilde{M}(m;n) = \text{span}\left\{(x-y)^{m+\frac{n}{2}-k} R_k^{m,n}\left(\frac{x+y}{x-y}\right) \quad , \quad 0 \le k \le 2m+n\right\} \tag{11}$$

$$R_k^{m,n}(t) = \frac{d^k}{dt^k}(t-1)^{m+n}(t+1)^m \tag{12}$$

Our observation is summarized by the following two formulas

$$(x-y)^{m+\frac{n}{2}}\ \tilde{j}_\epsilon\ (x-y)^{-m-\frac{n}{2}} = j_\epsilon(x,m) + j_\epsilon(y,m+n) \tag{13}$$

$$(x-y)^{m+\frac{n}{2}}\tilde{M}(m;n) = M(m;m+n) \tag{14}$$

In other words the algebra (10) is equivalent to the algebra (7) (up to a suitable redefinition of n into $m+n$).

The advantage of the new formulation is twofold. First the relevant operators form an sl(2) diagonal subalgebra of the sl(2)× sl(2) algebra generated by

$$j_\epsilon(x,m) \quad , \quad j_\epsilon(y,n) \quad , \quad \epsilon = \pm, 0 \tag{15}$$

In this respect the form (7) of the operators (10) is clearly related to the fundamental operators (4) and is easy to generalize to the case of M variables. Second, the vector space of the representation, i.e. $M(m;n)$, is a space of polynomials like in all the other cases of the classification of Ref.[5]. In the next sections, we discuss some possible extensions of the algebra (7) into graded algebras. The corresponding operators are related to systems of QES equations.

3 Superalgebra spl(M+1,N+1)

The bosonic part of the superalgebra spl(M+1,N+1) contains the Lie algebra sl(M+1)× sl(N+1) [9]. The fermionic generators split into two multiplets $(M+1,\overline{N+1})$ and $(\overline{M+1},N+1)$ under the adjoint action of the sl(M+1)× sl(N+1) subalgebra.

Following the results of [9], it appears that the typical representations of spl(M+1,N+1) can be constructed by applying suitable combinations of the fermionic operators to a given irreducible representation of sl(M+1)× sl(N+1), the so called ground floor. Then the whole representation is generated by applying the monomials in the fermionic on the ground floor. The set of vectors attained by means of a monomial of degree j defines the j^{th} floor. The anticommutation relations between the fermionic operators guarantee that the total number of floors is finite.

A natural vector space which can serve as a basis of representation of Lie algebra sl(M+1)× sl(N+1) is $P(m,M;n,N)$, defined above. Then, some irreducible representations of spl(M+1,N+1) can be constructed in terms of differential operators acting on a direct sum

$$\oplus_{k=1}^{K} P(m_k,M;n_k,N) \qquad (16)$$

The bosonic generators are the direct sums

$$\mathrm{diag}(J_a^b(\vec{x},m_1),\cdots,J_a^b(\vec{x},m_K)) \quad , \quad \mathrm{diag}(J_a^b(\vec{y},n_1),\cdots,J_a^b(\vec{y},n_K)) \qquad (17)$$

the fermionic ones are built with the tensorial operators obtained in [13]. They involve in general a large number of parameters to be fixed by imposing the commutation relations. The determination of the possible values for m_i, m_j, K and the explicit construction of the fermionic generators for generic values of M, N is a complicated task. We therefore limit our study to the particular cases spl(2,1) and spl(2,2). Our aim is to identify the hidden symmetries of quasi exactly solvable systems; the simplest of them are related to the operators involving a rather low number of variables and of polynomial components.

3.1 The superalgebra spl(2,1)

The bosonic part of spl(2,1), which is equivalent to the (perhaps better known) algebra osp(2,2), is the Lie algebra sl(2)× u(1). The structure constant appear rather simple when we label the bosonic generators J_a^b and the fermionic ones Q_a and $\overline{Q}^a$ $(a, b = 1, 2)$:

$$[\quad J_a^b, J_c^d] = \delta_a^d J_c^b - \delta_c^b J_a^d \tag{18}$$

$$[\quad J_a^b, Q_c] = -\delta_c^b Q_a + \delta_a^b Q_c \tag{19}$$

$$[\quad J_a^b, \overline{Q}^c] = \delta_a^c \overline{Q}^b - \delta_a^b \overline{Q}^c \tag{20}$$

$$\{\quad Q_a, \overline{Q}^b\} = J_a^b \tag{21}$$

$$\{\quad Q_a, Q_b\} = \{\overline{Q}^a, \overline{Q}^b\} = 0 \tag{22}$$

3.1.1 Representation by differential operators of one variable

The typical irreducible representations of spl(2,1) [16] can be expressed as 4×4 matrix differential operators acting on the 4-tuple of polynomials of one variable whose decomposition in floors reads

$$\text{floor } 0 \qquad P(m) \tag{23}$$

$$\text{floor } 1 \qquad P(m+1) \oplus P(m-1) \tag{24}$$

$$\text{floor } 2 \qquad P(m) \tag{25}$$

The operators

$$q_a(x) = (1, x) \quad ; \quad \overline{q}_a(x, m) = (\frac{d}{dx}, x\frac{d}{dx} - m) \tag{26}$$

which naturally connect $P(n)$ with $P(n\pm1)$ play a crucial role in the construction of the representation. Using the symplectic metric ϵ^{ab} with $\epsilon^{01} = 1$ to raise and lower the indices, the fermionic generators can be set in the form

$$Q_a = \begin{pmatrix} 0 & 0 & 0 & \\ q_a & 0 & 0 & 0 \\ \overline{q}_a(n) & 0 & 0 & 0 \\ 0 & \overline{q}_a(n+1) & -q_a & 0 \end{pmatrix} \tag{27}$$

$$\overline{Q}^a = \begin{pmatrix} 0 & \alpha\overline{q}^a(n+1) & \beta q^a & 0 \\ 0 & 0 & 0 & \alpha q^a \\ 0 & 0 & 0 & -\beta\overline{q}^a(n) \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

with

$$\alpha = \frac{n-t}{2(n+1)} \quad , \beta = \frac{n+2+t}{2(n+1)} \tag{29}$$

Where t is an arbitrary parameter and we made use of the similarity transformation to set Q_a in a particularly simple form. The bosonic operators read then

$$J_0^1 = -\mathrm{diag}(j_-(x), j_-(x), j_-(x), j_-(x)) \tag{30}$$

$$J_1^0 = \mathrm{diag}(j_+(x,m), j_+(x,m+1), j_+(x,m-1), j_+(x,m)) \tag{31}$$

$$\frac{1}{2}(J_0^0 + J_1^1) = \frac{1}{2}\mathrm{diag}(t, t+1, t+2, t+2) \tag{32}$$

$$\frac{1}{2}(J_0^0 - J_1^1) = \mathrm{diag}(j_0(x,m), j_0(x,m+1), j_0(x,m-1), j_0(x,m)) \tag{33}$$

The irreducibility of the representation (for generic value of t) and the use of Burnside theorem guarantee that the eight operators above generate all linear, differential operators preserving the space $P(m) \oplus P(M+1) \oplus P(M-1) \oplus P(m)$.

In the limit $t = -(m+2)$ (i.e. $\beta = 0$), the upper left 2×2 blocks of the operators above act invariantly on the space $P(m) \oplus P(m+1)$ and lead to a series of atypical representations. This formulation of the Lie superalgebra osp(2,2) by differential operators (of one variable) preserving $P(m) \oplus P(m+1)$ was first noticed in [6].

3.1.2 Representations by differential operators of two variables

By using the formalism of sect.2.1 one can also build representations of spl(2,1) in terms of differential operators of two (or more) variables. Let us first discuss the representation by operators preserving the direct sum

$$M(m;n) \oplus M(m+1;n) \quad . \tag{34}$$

The diagonal generators are of the form (see(6))

$$\mathrm{diag}(J_a^b(x,m) + J_a^b(y,n), J_a^b(x,m+1) + J_a^b(y,n)) - \delta_a^b \mathrm{diag}(1,0) \tag{35}$$

(see (7)), we have to construct the off diagonal ones which connect the two subspaces of the sum (34). In this case, the relevant tensorial operators read

$$\begin{aligned} q_0(x,m;y,n) &= \frac{1}{m+1}(m+n+1+(x-y)\frac{\partial}{\partial y}) \\ q_1(x,m;y,n) &= \frac{1}{m+1}\left((m+1)x+ny+y(x-y)\frac{\partial}{\partial y}\right) \end{aligned} \tag{36}$$

$$\begin{aligned} \overline{q}_0(x,m) &= \frac{\partial}{\partial x} \\ \overline{q}_1(x,m) &= (x\frac{\partial}{\partial x} - m) \ . \end{aligned} \tag{37}$$

They generalize (26) which are recovered by setting $n = 0$ and dropping all derivatives $\frac{\partial}{\partial y}$. We then obtain the desired representation in terms of (35), supplemented by the fermionic generators

$$Q_a = q_a(x,m;y,n)\sigma_- \quad , \overline{Q}^a = \overline{q}^a(x,m)\sigma_+ \qquad , \quad a = 0,1 \tag{38}$$

It has dimension $2m+2n+3$; again Burnside theorem guarantees that all differential, linear operators preserving (34) are the elements of the envelopping algebra generated by the eight generators.

The generic typical representations of spl(2,2) can also be formulated in terms of the two variables differential operators. They can be constructed equally well on the vector spaces

$$V_I = M_{m,n} \oplus M_{m+1,n} \oplus M_{m-1,n} \oplus M_{m,n} \tag{39}$$

$$V_{II} = M_{m,n} \oplus M_{m+1,n} \oplus M_{m,n+1} \oplus M_{m,n} \tag{40}$$

The associated representations are equivalent (in agreement with [16]) but the expressions of the generators in terms of the partial derivatives is quite different. The proof of the equivalence between the representations acting on V_I and on V_{II} relies on the fact that the operators (35),(37) are invariant under the double substitution $m \leftrightarrow n$ and $x \leftrightarrow y$. Therefore the operators $q_a(y,n;x,m)$ (resp. $\overline{q}^a(y,n)$) behave exactly as $q_a(x,m;y,n)$ (resp. $\overline{q}^a(x,m)$) under the adjoint action of gl(2) algebra (35).

3.2 The superalgebra spl(2,2)

The bosonic part of the superalgebra spl(2,2) is sl(2)× sl(2)× u(1) [9]. We note $J_a^b, \tilde{J}_a^b$ the two sets of generators of the two sl(2) ($\sum_a J_a^a = \sum_a \tilde{J}_a^a = 0$) and Y the generator related to $u(1)$. Denoting the fermionic generators by $Q_a^b, \overline{Q}_a^b$ the commutator relations read as follow

$$[\quad J_a^b, J_c^d] = \delta_a^d J_c^b - \delta_e^b J_a^d \tag{41}$$

$$[\quad \tilde{J}_a^b, \tilde{J}_c^d] = \delta_a^d \tilde{J}_c^b - \delta_c^b \tilde{J}_a^d \tag{42}$$

$$[\quad J_a^b, \tilde{J}_c^d] = [J_a^b, Y] = [\tilde{J}_a^b, Y] = 0 \tag{43}$$

$$[\quad J_a^b, Q_c^d] = -\delta_c^b Q_a^d + \frac{1}{2}\delta_a^b Q_c^d \tag{44}$$

$$[\quad J_a^b, \overline{Q}_c^d] = \delta_a^d \overline{Q}_c^b - \frac{1}{2}\delta_a^b \overline{Q}_c^d \tag{45}$$

$$[\ \tilde{J}^b_a, Q^d_c] = \delta^d_a Q^b_c - \frac{1}{2}\delta^b_a Q^d_c \tag{46}$$

$$[\ \tilde{J}^b_a, \overline{Q}^d_c] = -\delta^b_c \overline{Q}^d_a + \frac{1}{2}\delta^b_a \overline{Q}^d_c \tag{47}$$

$$\{\ Q^b_a, \overline{Q}^d_c\} = \delta^b_c J^d_a + \delta^d_a J^d_c + \frac{1}{2}\delta^d_a \delta^b_c Y \tag{48}$$

$$\{\ Q^b_a, Q^d_c\} = \{\overline{Q}^b_a, \overline{Q}^d_c\} = 0 \tag{49}$$

The structure of the anticommutators $\{Q, Q\}$ is such that the generic representation consist of five floors [9]. In term of polynomial space of two variables $P(m;n)$ the representation is organized as follow

$$\begin{array}{ll} \text{floor } 0 & P(m;n) \\ \text{floor } 1 & P(m+1;n+1) \oplus P(m+1;n-1) \\ & \oplus P(m-1;n+1) \oplus P(m-1;n-1) \\ \text{floor } 2 & P(m+1;n) \oplus P(m;n+2) \oplus P(m;n) \\ & \oplus P(m;n) \oplus P(m;n-2) \oplus P(m-2;n) \\ \text{floor } 3 & P(m+1;n+1) \oplus P(m+1;n-1) \\ & \oplus P(m-1;n+1) \oplus P(m-1;n-1) \\ \text{floor } 4 & P(m;n) \end{array} \tag{50}$$

The construction of the fermionic generators involves the determination of 112 parameters. Like for the case spl(2,1) we focused our attention on the representations composed out of a lower number of levels. If we restrict to two levels only, we find [9] that the only possibility is the space $P(1;0) \oplus P(0;1)$, correspondingly $Y = 1$. It was already observed in [13] that the set of operators preserving the vector space $P(m;n) \oplus P(k;l)$ do not represent a linear algebra. Considering then the representations with three levels, we found a series of atypical representations with the following dimensions

$$\begin{array}{ll} \text{floor } 0 & P(n,n) \\ \text{floor } 1 & P(n+1,n) \oplus P(n-1,n+1) \end{array}$$

$$\text{floor 2} \qquad P(n,n) \tag{51}$$

The bosonic operators read as follows

$$\begin{aligned}
J_0^1 &= -\mathrm{diag}(J_-(x), J_-(x), J_-(x), J_-(x)) \\
J_0^0 &= \mathrm{diag}(J_0(x,n), J_0(x,n+1), J_0(x,n-1), J_0(x,n) \\
J_1^0 &= \mathrm{diag}(J_+(x,n), J_+(x,n+1), J_+(x,n-1), J_+(x,n)) \\
\tilde{J}_0^1 &= -\mathrm{diag}(J_-(y), J_-(y), J_-(y), J_-(y)) \\
\tilde{J}_0^0 &= \mathrm{diag}(J_0(y,n), J_0(y,n-1), J_0(y,n+1), J_0(y,n)) \\
\tilde{J}_0^1 &= \mathrm{diag}(J_+(y,n), J_+(y,n-1), J_+(y,n+1), J_+(y,n))
\end{aligned} \tag{52}$$

and the value Y is just vanishing for the representation under consideration. Finally, the fermionic operators are of the form

$$Q_a^b = \epsilon^{bc} V_{ac} \quad , \quad \overline{Q}_a^b = \epsilon^{bc} \overline{V}_{ca} \tag{53}$$

with

$$V_{ac} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q_a \overline{q}_c(n) & 0 & 0 & 0 \\ \overline{q}_a(n) q_c & 0 & 0 & 0 \\ 0 & \overline{q}_a(n+1) q_c & -q_a \overline{q}_c(n+1) & 0 \end{pmatrix} \tag{54}$$

$$\overline{V}_{ac} = \frac{1}{n+1} \begin{pmatrix} 0 & -\overline{q}_a(n+1) q_c & q_a \overline{q}_c(n) & 0 \\ 0 & 0 & 0 & q_a \overline{q}_c(n) \\ 0 & 0 & 0 & \overline{q}_a(n) q_c \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{55}$$

where, for shortness, we dropped the variable x (resp. y) for the operator $q_a, \overline{q}_a$ (resp. $q_c, \overline{q}_c$). Again, a suitable similarity transformation was used to set V_{ac} in a form a simple as possible.

4 More graded algebras

The four operators (26) play a crucial role in the construction of the operators preserving the spaces $P(m) \oplus P(m+\Delta)$ [7, 8] and therefore in the classification of 2×2 QES systems. It is known that the underlying algebra is non linear if $\Delta > 1$. In this section we construct the operators preserving the vector space

$$M(m;n) \oplus M(m+\Delta;n+\Delta') \qquad . \tag{56}$$

The relevant diagonal operators are conveniently chosen according to (6)

$$\begin{aligned} & \mathrm{diag}(J_a^b(x,m) + J_a^b(y,n), J_a^b(x,m+\Delta) + J_a^b(y,n+\Delta')) \\ - & \frac{1}{2}\delta_a^b \mathrm{diag}(1+\Delta+\Delta', 1-\Delta-\Delta') \end{aligned} \tag{57}$$

The construction of the off diagonal ones depends on the relative signs of Δ and Δ'.

4.1 case $\Delta, \Delta' \geq 0$

The operators connecting the vector space $M(m;n)$ with $M(m+\Delta;n+\Delta')$ (and vice versa) can be formulated in terms of products of operators (37) where the indices m and n in the different factors are appropriately shifted . The following identities allows one to deal with the ambiguities of ordering of the different factors:

$$\begin{aligned} q_b(x,m+1;y,n)\ q_a(x,m;y,n) &= q_a(x,m+1;y,n)q_b(x,m;y,n) \\ q_b(y,n;x,m+1)\ q_a(x,m;y,n) &= q_a(y,n;x,m+1)q_b(x,m;y,n) \\ q_b(y,n;x,m+1)\ q_a(x,m;y,n) &= q_a(x,m;y,n+1)q_b(y,n;x,n) \end{aligned} \tag{58}$$

One can use these identities in order to define the operators

$$q(x,[a_k];y,[b_l]) \equiv \prod_{l=1}^{\Delta'}\prod_{k=1}^{\Delta} q_{b_l}(y,n+l-1;x,m+\Delta)q_{a_k}(x,m+k-1,n) \tag{59}$$

symmetrically in the multi indices

$$[a_k] \equiv (a_1, \ldots, a_{\Delta'}) \quad , \qquad a_i = 0, 1$$
$$[b_l] \equiv (b_1, \ldots, b_{\Delta}) \quad , \qquad b_i = 0, 1 \tag{60}$$

The operators (59) connect $M(m;n)$ with $M(m+\Delta; n+\Delta')$ and

$$Q([a_k],[b_l]) = q(x,[a_k];y,[b_l])\sigma_- \tag{61}$$

are the counterparts of the operators (36). Using the remarks made at the end of the previous section, one can see that the operators (59) transform according to the representation of spin $\Delta+\Delta'$ under the adjoint action of the gl(2) represented via (6).

The operators $\overline{Q}([a_k],[b_l])$, proportional to σ_+, can be constructed in exactly the similar way as for the $Q([a_k],[b_l])$. Identities like (58) exist among the $\overline{q}_a$. The complete algebra (which is non linear) can be obtained following the same lines as in [8]. The anticommutator $\{Q, \overline{Q}\}$ close as polynomial expressions of (6) provided the constants γ are choosen such that

$$J_a^a = \mathrm{diag}(m+n+\Delta+\Delta'+1, m+n+1) \tag{62}$$

4.2 The case M(m;n) $\oplus$ M(m+1;n-1)

If we consider Δ and Δ' of opposite signs, the operators preserving (5) do not represent a Lie super algebra even for $|\Delta| = |\Delta'| = 1$ We studied the operators preserving the vector space $M(m;n) \oplus M(m+1;n-1)$ and observed that the underlying algebric stucture is different from those obtained in [8]. Again, $J_a^b(x,m,m+1;y,n,n-1)$ can be used as a starting point. We find it convenient to set $T=0$ in (6) and add separately the grading operator $T \equiv \sigma_3$. As far as the off diagonal operators are concerned, we choose

$$\begin{aligned} R_a^b &= \overline{q}_a(y,n-1,x,m+1)q^b(x,m,y,n)\sigma_- \\ \overline{R}_a^b &= \overline{q}_a(x,m,y,n)q^b(y,n-1,x,m+1)\sigma_+ \end{aligned} \tag{63}$$

These tensors are not irreducible with respect to the adjoint action of the J generators. The traces

$$R^a_a = \frac{m+n+2}{m+1}((y-x)\frac{\partial}{\partial y} - n) \quad , \quad \overline{R}^a_a = \frac{m+n+2}{n+1}((x-y)\frac{\partial}{\partial x} - m) \quad (64)$$

are operators which intertwine the equivalent representations carried by the spaces $M(m;n)$ and $M(m+1;n-1)$; they commute with the four operators (6).

The order of the factors q and $\overline{q}$ entering in R and $\overline{R}$ can be reversed by using the identity

$$\begin{aligned} q_a(y,n-1,x,m+1)q^b(x,m,y,n) &- q^b(x,m,y,n-1)q_a(y,n-1,x,m) \\ &= \frac{1}{m+n+2}R^a_a \end{aligned} \quad (65)$$

The generators $J^b_a, R^b_a, \overline{J}^b_a$ obey the following commutation and anticommutation relations

$$\begin{aligned} [J^b_a, R^d_c] &= \delta^d_a R^b_c - \delta^b_c R^d_a \ , \\ [J^b_a, \overline{R}^d_c] &= \delta^d_a \overline{R}^b_c - \delta^b_c \overline{R}^d_a \end{aligned} \quad (66)$$

$$\begin{aligned} \{R^b_a, \overline{R}^d_c\} &= \frac{1}{2}\{J^d_a, J^b_c\} + \frac{T}{2}(\delta^d_a J^b_c - \delta^b_c J^d_a) \\ &- \frac{1}{2}(\delta^b_a J^d_c + \delta^d_c J^b_a) - \frac{1}{2}\delta^b_a \delta^d_c \end{aligned} \quad (67)$$

These relations do not define an abstract algebra. In order to fulfil the associativity conditions (i.e. the Jacobi identities), the anticommutators between two R (two $\overline{R}$), which vanish for (63) have to be implemented in a non trivial way [7],[8]; equation (67) indicates that the underlying algebra is non linear.

5 Deformation of spl(2,1)

If we want to describe in abstract terms the algebraic structure underlying the QES finite difference equations, some deformations of the algebras discussed

above seem to emerge in a natural way. For scalar equations the relevant deformation was pointed out some time ago [3]; it is related to the “Witten type II” deformation of sl(2) [10, 12] Since the most relevant examples of QES systems are related to spl(2,1), it is natural to try to construct deformations of spl(2,1) whose representations can be formulated in terms of finite difference operators.

One deformation of spl(2,1) was obtained in [17] (for more general graded algebra see [18]); This deformation is such that the commutators of some generators close into transcendental functions of other generators. In a previous paper [13] some representations of this deformation of spl(2,1) were formulated in terms of finite difference operators.

More recently [14] a family of quommutator deformations of the superalgebra spl(p,1) was constructed. The quommutators and anti-quommutators are defined respectively as

$$[A, B]_q = AB - qBA \quad , \quad \{A, B\}_q = AB + qBA \tag{68}$$

where q is the deformation parameter. The quommutator deformations are closer, in the spirit, to the Witten type II deformation of sl(2). They lead in particular to natural ordering rules on the generators. In contrast to the deformation [17], exchanging the order of quommutating generators can be performed while keeping the polynomial character of the expression.

Here we present with the tensor notations the deformation of spl(2,1) which appears to be the most relevant for QES difference equations. It is expressed as follows, $(\mu, \nu, \alpha \ldots = 0, 1)$

$$\{Q_\mu, Q_\nu\}_{q^{\nu-\mu}} = 0 \quad , \quad \{\overline{Q}^\mu, \overline{Q}^\nu\}_{q^{\nu-\mu}} = 0 \tag{69}$$

$$\{Q_\mu, \overline{Q}^\nu\} = J_\mu^\nu \tag{70}$$

$$[J_\mu^\nu, Q_\alpha]_{q^{\alpha-\nu}} = q^{\frac{\alpha-\nu-1}{2}} (\delta_\mu^\nu Q_\alpha - \delta_\alpha^\nu Q_\mu) \tag{71}$$

$$[J_\mu^\nu, \overline{Q}^\alpha]_{q^{\mu-\alpha}} = q^{\frac{\mu-\alpha-1}{2}} (\delta_\mu^\alpha \overline{Q}^\nu - \delta_\mu^\nu \overline{Q}^\alpha) \tag{72}$$

$$
\begin{aligned}
[J^\nu_\mu, J^\beta_\alpha]_{q^s} &= q^{\frac{s-r-1}{2}}(\delta^\beta_\mu J^\nu_\alpha - q^r \delta^\nu_\alpha J^\beta_\mu) \\
&+ \frac{q-1}{q^2}(Q_0\overline{Q}^0 + Q_1\overline{Q}^1)\delta^\nu_\alpha\delta^\beta_\mu(1-\delta^\nu_\mu\delta^\beta_\alpha) \qquad (73)
\end{aligned}
$$

with $s \equiv \nu + \alpha - \mu - \beta$ and $r \equiv (\nu - \beta)(\mu - \alpha)$.

All Jacobi identities are obeyed by these relations. The last commutator indicates that (like for the deformation [17]) the bosonic generators do not close into a gl(2) subalgebra . Neglecting all fermionic operators in the above formulas leads to one deformation of gl(2) found in [19].

It is possible to construct two independent expressions, quadratic in the generators, which q-commute with the generators. These "q-Casimir" operators read

$$
\begin{aligned}
C_1 = & \; Q_0\overline{Q}^0 + Q_1\overline{Q}^1 + qJ^0_1 J^1_0 - J^0_0 J^1_1 - J^0_0 \\
C_2 = & \; (q-1)^2(Q_0\overline{Q}^0 + Q_1\overline{Q}^1) \\
& + q(q-1)^2 J^0_1 J^1_0 + (q-1)J^1_1 - q(q-1)J^0_0 - 1 \qquad (74)
\end{aligned}
$$

and obey the following commutation properties (for i=1,2)

$$
\begin{aligned}
[C_i, J^\nu_\mu]_{q^{2(\mu-\nu)}} &= 0 \quad , \\
[C_i, Q_\mu]_{q^{2\mu-1}} &= 0 \quad , \\
[C_i, Q^\mu]_{q^{1-2\mu}} &= 0 \qquad (75)
\end{aligned}
$$

It follows that any function of the ratio C_1/C_2 commute with the generators and constitute a conventional Casimir.

We further constructed the representations of the algebra above which are relevant for systems of finite difference QES equations. To describe them we define a finite difference operator D_q.

$$
D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad , \quad D_q x^n = [n]_q x^{n-1} \quad , \quad [n]_q \equiv \frac{1-q^n}{1-q} \qquad (76)
$$

The simplest of these realizations are characterized by a positive integer n and act on the vector space

$$
P(n-1) \oplus P(n) \quad . \qquad (77)
$$

Adopting x as the variable, the fermionic generators are represented by

$$\begin{aligned} Q_0 &= q^{-\frac{n}{2}}\sigma_- \quad , \qquad & Q_1 &= -x\sigma_- \\ \overline{Q}^0 &= q^{-\frac{n}{2}}(xD_q - [n]_q)\sigma_+ \quad , \qquad & \overline{Q}^1 &= D_q\sigma_+ \end{aligned} \tag{78}$$

The bosonic operators can be constructed easily from (70) but we write them for completeness

$$J_0^0 = q^{-n}((xD_q - [n]_q)\mathbb{I}_2 \quad , \quad J_1^1 = (-1)\mathrm{diag}(qxD_q + 1, xD_q) \tag{79}$$

$$J_0^1 = q^{-\frac{n}{2}} D_q \mathbb{I}_2 \quad , \quad J_1^0 = (-1)q^{-\frac{n}{2}}\mathrm{diag}(qx(xD_q - [n-1]_q), x(xD_q - [n]_q)) \tag{80}$$

The enveloping algebra constructed with the eight generators above contains all finite difference operators preserving $P(n-1) \oplus P(n)$.

As in the underformed case, there also exist representations which act on the vector space

$$P(n) \oplus P(n+1) \oplus P(n-1) \oplus P(n) \tag{81}$$

The fermionic generators are realized as follows

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ D_q & 0 & 0 & 0 \\ 0 & -D_q & 1 & 0 \end{pmatrix} \tag{82}$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ q^{-n}\delta(n) & 0 & 0 & 0 \\ 0 & -q^{-n-1}\delta(n+1) & x & 0 \end{pmatrix} \tag{83}$$

$$\overline{Q}^0 = \begin{pmatrix} 0 & \lambda\delta(n+1) & (1-\lambda q^{n+1})x & 0 \\ 0 & 0 & 0 & (\lambda q^{n+1} - 1)x \\ 0 & 0 & 0 & q\lambda\delta(n) \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{84}$$

$$\overline{Q}^1 = \begin{pmatrix} 0 & \lambda q^{n+1}D_q & (1-\lambda q^{n+1}) & 0 \\ 0 & 0 & 0 & \lambda q^{n+1}-1 \\ 0 & 0 & 0 & \lambda q^{n+1}D_q \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (85)$$

where λ is an arbitrary complex parameter and, for shortness, we used $\delta(n) \equiv xD_q - [n]_q$. Once more, the invariance under the similarity transformations has been exploited to set Q_0 in a form as simple as possible. In the limit $\lambda \to q^{-n-1}$ the representation (85) becomes reducible and decomposes into atypical ones of the form (78).

The deformation of spl(2,1) presented above leads to simple normal ordering rules for the generators. In this respect, it is very appropriate for the classification of the finite difference operators preserving the space (77) or (81) and of the corresponding QES systems. The normal ordering rules associated with the deformation used in ref.[13] are not as transparent since they depend non polynomially of some operators.

6 Concluding remarks

The most interesting examples of QES systems are related to the algebra spl(2,1), e.g. the relativistic Coulomb problem and the stability of the sphaleron in the abelian Higgs model [7]. Very recently, the hidden algebras of the supersymmetric versions of the Calogero and Sutherland models were shown to be the superalgebras spl(N+1,N) [21].

Further realizations of these algebras, and more generally of the superalgebras spl(p,q), in terms of differential operators therefore desserve some attention. Here, we present realizations of spl(2,1) formulated in terms of differential operators in two variables; extensions to the case of an arbitrary number of variables can be achieved in a straightforward way. We also constructed a series of atypical representations of spl(2,2) in terms of operators acting on a set of 4-tuple of

polynomials in two variables. The labelling used for the generators clearly exhibits their tensorial structure under their bosonic subalgebra and provides very naturally the building blocks for the construction of series of (non linear) graded algebras preserving more general vector spaces of polynomials.

Witten type deformations attracted recently some attention (see e.g. [20] for osp(1,2)). The deformation of spl(2,1) presented here is of this type and it admits representations which are directly relevant for the study of QES finite difference systems. The correspondance between discrete operators and finite difference one is discussed in [22] where also some some applications in discrete quantum mechanics are pointed out.

The existence of a coproduct for this type of deformation would allow to adapt the construction of section 2 to finite difference equations. This is nice an application of the coproduct that we plan to address later.

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